Initial Uncertainties in the EPS: Singular Vector Perturbations

Training Course 2015

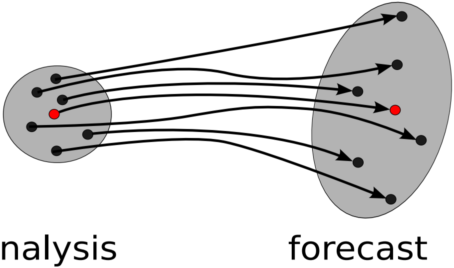
An evolving EPS

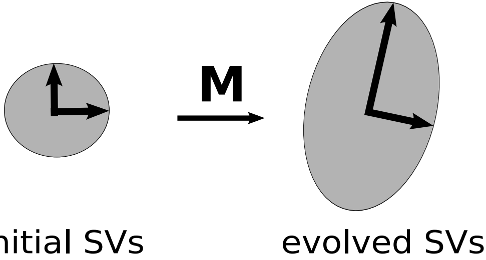
- EPS 1992–2010: initial perturbations based on singular vectors (SVs)
- Are SVs optimal?
- Ideally, SVs should be computed with an initial time norm based on the analysis uncertainty of the day.
- However, if a good estimate of the analysis uncertainty of the day is available, SVs will not be required any more for the initial perturbations.
- The goal is to obtain a sample of the distribution of initial uncertainty from an ensemble of data assimilations (see Roberto's talk)

Outline of this lecture

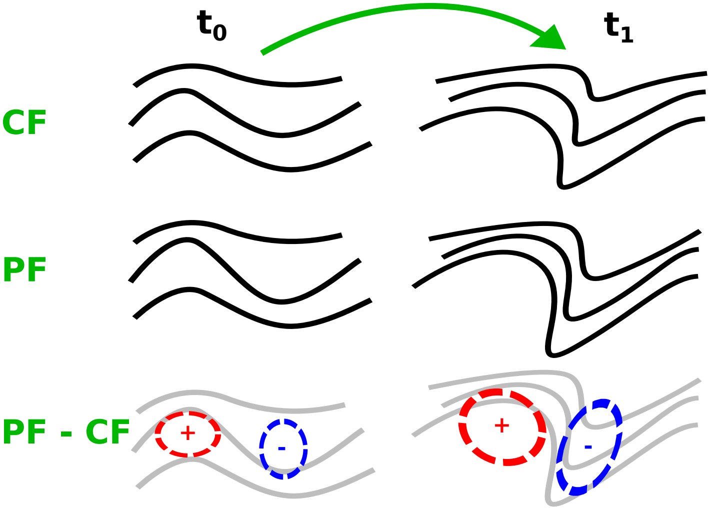
1. **singular vectors?**
2. **perturbations?**
3. **some background:**
   - perturbation growth etc.
   - norms
   - singular value decomposition
   - tangent-linear system
4. **an idealised example:** singular vectors in the Eady model
5. **SVs in the operational EPS**
6. **initial condition perturbations**

Singular Vectors?





Perturbations?



Perturbation Dynamics (I)

see Kalnay (2003)

$$\frac{d\mathbf{x_r}}{dt} = F(\mathbf{x_r}) \quad \text{with} \quad \mathbf{x_r} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$$

where $\mathbf{x_r} \in \mathbb{R}^N$ denotes the N-dimensional state vector and $F(\mathbf{x_r}) \in \mathbb{R}^N$ its tendency.

Integrate from t_0 to t gives the nonlinear model:

$$\mathbf{x_r}(t) = \mathcal{F}(\mathbf{x_r}(t_0))$$

Perturbation Dynamics (II)

see Kalnay (2003)

Taylor expansion:

$$\mathcal{F}(\mathbf{x_r}(t_0)+\mathbf{x}(t_0)) = \mathcal{F}(\mathbf{x_r}(t_0)) + \frac{\partial \mathcal{F}}{\partial \mathbf{x_r}}\mathbf{x}(t_0) + \mathcal{O}(\mathbf{x}(t_0)^2) + \dots$$

with $\mathbf{x}$ small perturbation to $\mathbf{x_r}$.

Neglect higher order terms:

$$\mathbf{x_r}(t)+\mathbf{x}(t) \approx \mathcal{F}(\mathbf{x_r}(t_0)) + \mathbf{M}_{[t_0,t]}\mathbf{x}(t_0)$$

here, $\mathbf{M}_{[t_0,t]}$ is the tangent linear propagator from t_0 to t. $\mathbf{M}_{[t_0,t]}$ evolves perturbations from t_0 to t:

$$\mathbf{x}(t) = \mathbf{M}_{[t_0,t]}\mathbf{x}(t_0)$$

Perturbation Growth

Perturbation growth is defined as:

$$\sigma^2 = \frac{\langle \mathbf{x}(t), \mathbf{x}(t) \rangle}{\langle \mathbf{x}(t_0), \mathbf{x}(t_0) \rangle}$$

$$= \frac{\langle \mathbf{M}_{[t_0,t]}\mathbf{x}(t_0), \mathbf{M}_{[t_0,t]}\mathbf{x}(t_0) \rangle}{\langle \mathbf{x}(t_0), \mathbf{x}(t_0) \rangle}$$

$$= \frac{\langle \mathbf{M}^T_{[t_0,t]}\mathbf{M}_{[t_0,t]}\mathbf{x}(t_0), \mathbf{x}(t_0) \rangle}{\langle \mathbf{x}(t_0), \mathbf{x}(t_0) \rangle}$$

with inner product $\langle \cdot, \cdot \rangle$ and growth factor σ^2.

$\Rightarrow$ Largest growth is associated with eigenvectors of $\mathbf{M}^T_{[t_0,t]}\mathbf{M}_{[t_0,t]}$. These eigenvectors are determined by a singular value decomposition of $\mathbf{M}_{[t_0,t]}$.

Norms

- The definition of singular vectors in the context of ensemble prediction involves norms (based on an inner product or metric). These are required to measure the amplitude of perturbations.

$$\langle \mathbf{x}, \mathbf{x} \rangle_C = \mathbf{x}^T \mathbf{C} \mathbf{x}$$

where $\mathbf{C}$ is symmetric ($\mathbf{C}^{\mathrm{T}} = \mathbf{C}$) and positive definite ($\mathbf{x}^{\mathrm{T}} \mathbf{C} \mathbf{x} > 0$ for $\mathbf{x} \neq 0$).

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- For predictability applications, the appropriate choice for the **initial time norm** is the analysis error covariance metric, *i.e.* the norm that is based on the inverse of the initial error covariance matrix (or some estimate thereof).

$$\|\mathbf{x}\|_i^2 = \mathbf{x}^{\mathrm{T}} \mathbf{C}_0^{-1} \mathbf{x}$$

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- The **final time norm** $\|\mathbf{x}\|_f$ is a convenient RMS measure of forecast error.
- **Total energy norm** is used both at initial and final time for the operational singular vector computations at ECMWF:

$$\|\mathbf{x}\|_{\mathbf{E}}^2 = \mathbf{x}^{\mathrm{T}}\mathbf{E}\mathbf{x} = \frac{1}{2}\int_{p_0}^{p_1}\int_S \left(u^2 + v^2 + \frac{c_p}{T_r}T^2\right) \mathrm{d}p\, \mathrm{d}s + \frac{1}{2} R_d T_r p_r \int_S (\ln p_{\mathrm{sfc}})^2 \, \mathrm{d}s$$

On the choice of the initial time norm

- The structure of singular vectors depends on the choice of the norm, in particular the initial time norm.
- An enstrophy norm at initial time penalises perturbations with small spatial scales, the initial SVs are planetary-scale structures.
- A streamfunction variance norm at initial time penalises the large scales and favours sub-synoptic scale perturbations.
- With a total energy norm at initial time, the energy spectrum of the initial SVs is “white” and best matches the spectrum of analysis error estimates from analyses differences (Palmer et al. 1998)

singular value decomposition of a matrix

Consider a matrix $$\mathbf{Q} = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Its singular value decomposition is defined as

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where **U** and **V** are orthogonal m-by-m and n-by-n matrices. Matrix **S** is a diagonal m-by-n matrix ($s_{ij} = 0$ if $i \neq j$, $s_{jj} \equiv \sigma_j$). The values σ_j on the diagonal of **S** are called *singular values*.

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The columns $\mathbf{u}_j$ of $\mathbf{U}$ are referred to as *left singular vectors* and the columns $\mathbf{v}_j$ of $\mathbf{V}$ are referred to as *right singular vectors*.
Eq. (1) implies that

$$\mathbf{Q}\,\mathbf{v}_j = \sigma_j \mathbf{u}_j$$

One can show that the $\mathbf{v}_j$ are the eigenvectors of $\mathbf{Q}^T\mathbf{Q}$!

see Golub and Van Loan: *Matrix Computations* for further details

singular value decomposition of the propagator

$$\mathbf{M} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^{\mathrm{T}} \quad \rightarrow \quad \mathbf{M}\,\mathbf{v}_j = \sigma_j \mathbf{u}_j$$

with the (initial) singular vectors $\mathbf{v}_j$ being the eigenvectors and the squared singular values σ_j^2 being the eigenvalues of $\mathbf{M}^T\mathbf{M}$. The $\mathbf{u}_j$ are called the evolved singular vectors.

Singular vectors are optimal perturbations in the following sense.

- the ratio of the final time norm to the initial time norm is given by the singular value:

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- Singular vector j is the direction in phase space that maximises the ratio of norms in the subspace orthogonal (with respect to $\mathbf{C}_0^{-1}$) to the space spanned by singular vectors $1 \ldots j-1$.

The tangent-linear model and its adjoint

- For a numerical model with $\sim 10^5 - 10^8$ variables it is not possible to obtain the propagator $\mathbf{M}$ as a matrix.

The tangent-linear model and its adjoint

• For a numerical model with $\sim 10^5 - 10^8$ variables it is not possible to obtain the propagator **M** as a matrix.

• Instead *algorithmic differentiation* is used to obtain the first derivative of the numerical algorithm that represents the forecast model.

For any initial perturbation **x**, the evolved perturbation **Mx** is obtained via an integration of the *tangent-linear model*.

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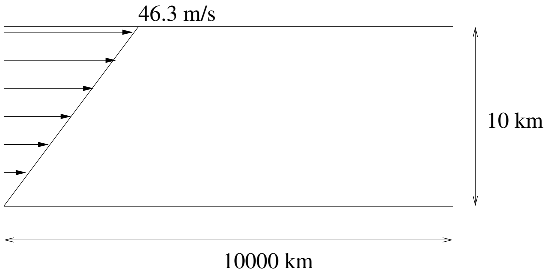
See also related presentations by Angela Benedetti, Marta Janisková and the "Hands on: Coding of Tangent Linear and Adjoints" session in Training Course on Data Assimilation & Use of Satellite Data

SVs in the Eady model

- channel with periodic boundary conditions in the zonal direction
- linear shear of basic state flow $\overline{U} = Sz$



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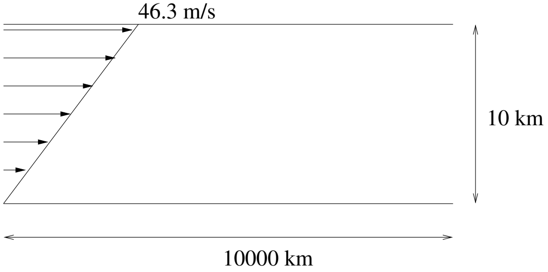
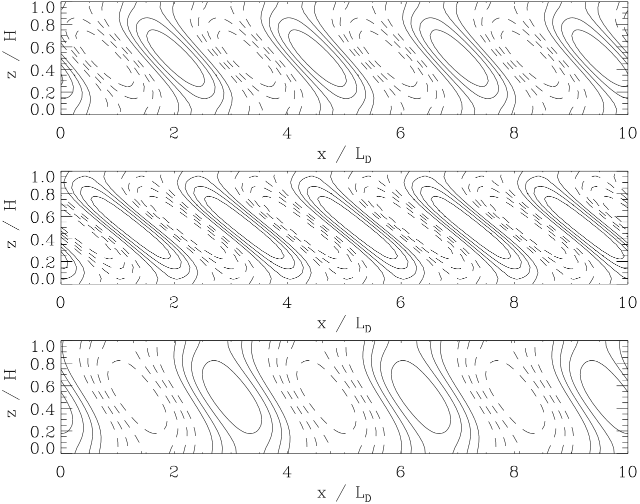


- f-plane with $f = 10^{-4}\,\mathrm{s}^{-1}$
- Brunt-Vaisala frequency $N = 10^{-2}\,\mathrm{s}^{-1}$
- total energy norm at initial and final time
- discretisation: 21 levels in the vertical , 16 wavenumbers in the horizontal

SVs in the Eady model: $t_{\text{opt}} = 24$ h

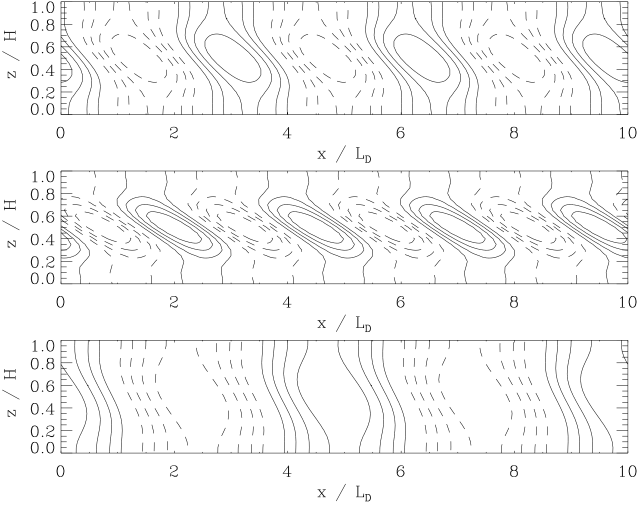
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 6.4, \quad \sigma_2 = 6.2, \quad \sigma_3 = 6.1.$

SVs in the Eady model: $t_{opt} = 48$ h

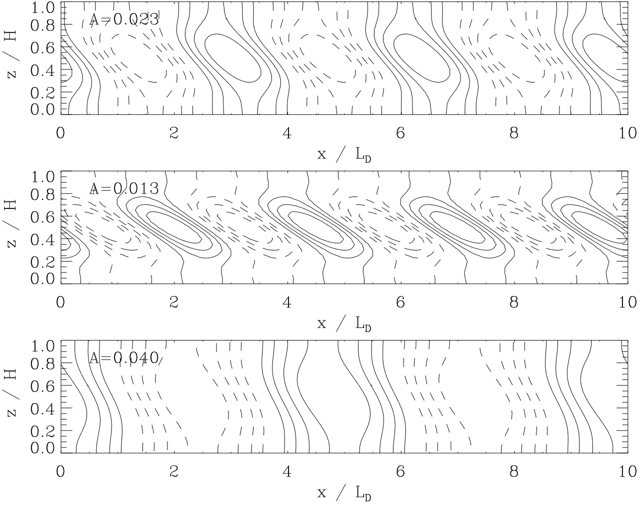
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 24.4, \quad \sigma_2 = 22.3, \quad \sigma_3 = 17.9.$

Eady model: $t_{\mathrm{opt}} = 48$ h, $\quad t = 0$ h

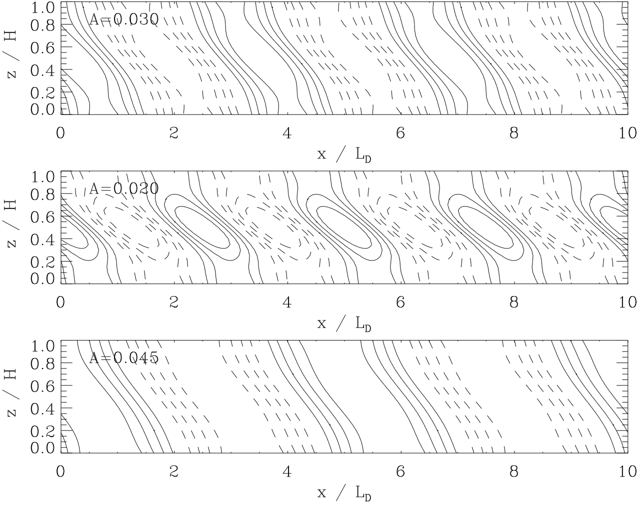
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 24.4, \quad \sigma_2 = 22.3, \quad \sigma_3 = 17.9.$

Eady model: $t_{\text{opt}} = 48$ h, $t = 6$ h

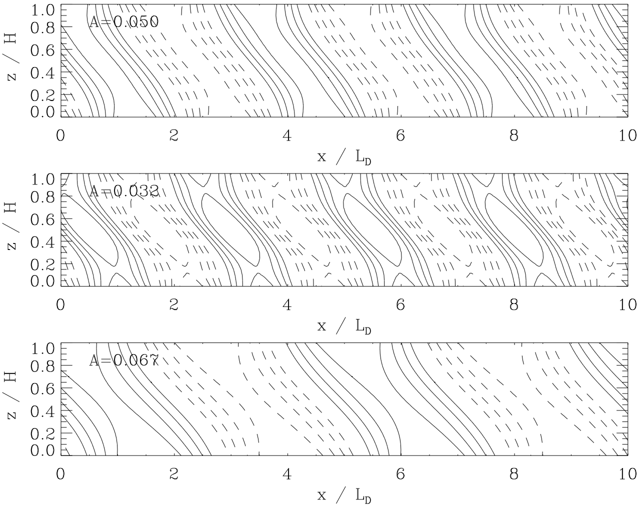
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 24.4$, $\sigma_2 = 22.3$, $\sigma_3 = 17.9$.

Eady model: $t_{\text{opt}} = 48$ h, $\quad t = 12$ h

streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



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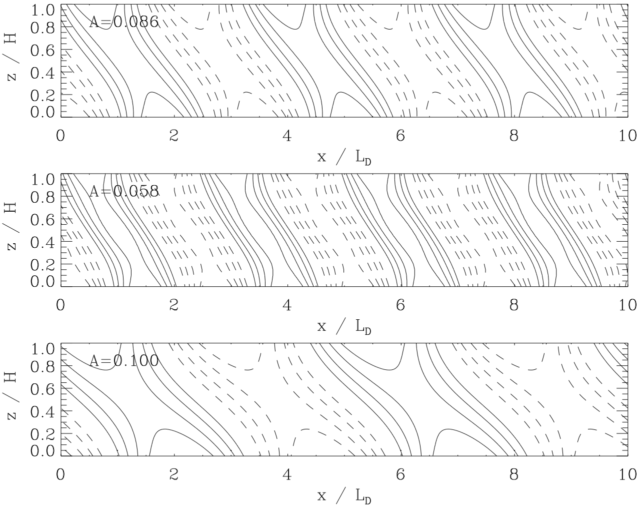
Eady model: $t_{\text{opt}} = 48$ h, $t = 18$ h

streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



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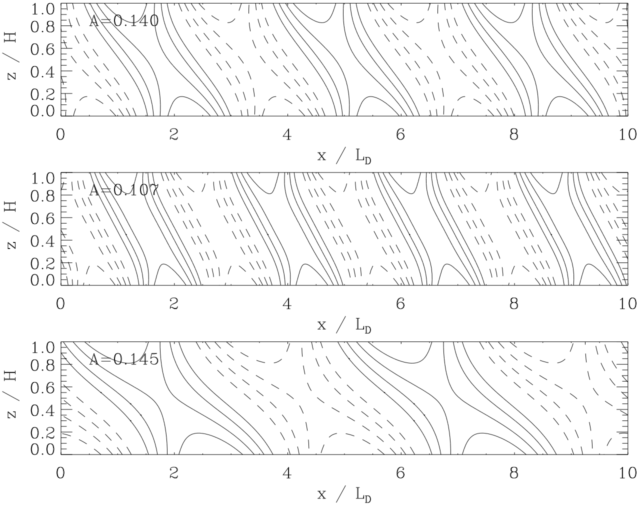
Eady model: $t_{\mathrm{opt}} = 48$ h, $t = 24$ h

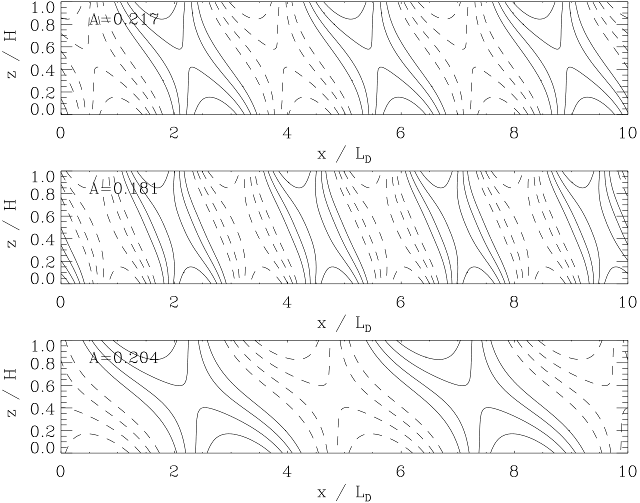
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 24.4$, $\sigma_2 = 22.3$, $\sigma_3 = 17.9$.

Eady model: $t_{\text{opt}} = 48$ h, $t = 30$ h

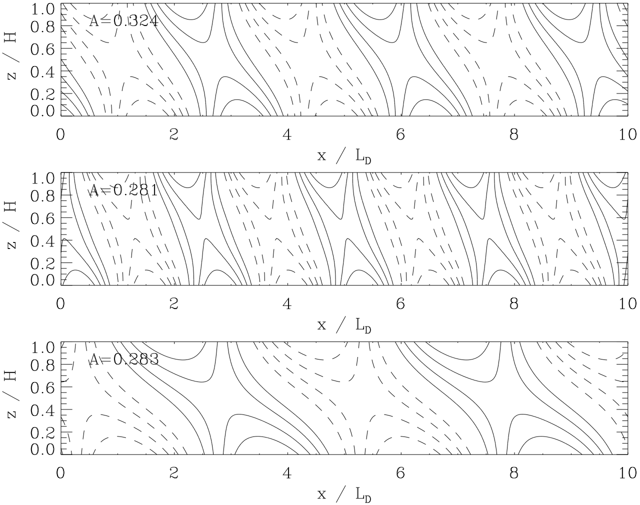
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 24.4, \quad \sigma_2 = 22.3, \quad \sigma_3 = 17.9.$

Eady model: $t_{\text{opt}} = 48$ h, $t = 36$ h

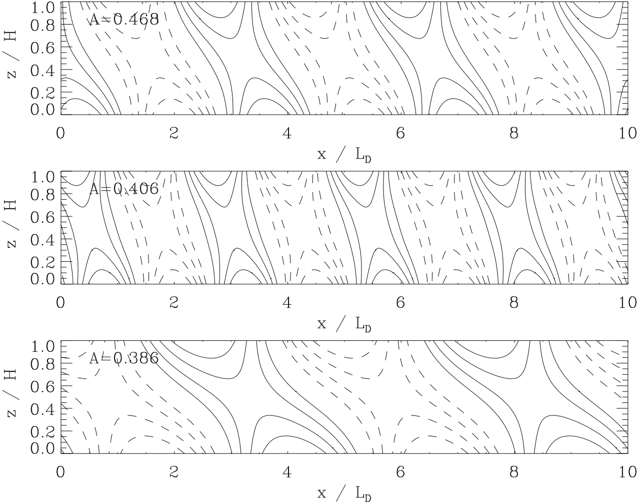
streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



singular values: $\sigma_1 = 24.4$, $\sigma_2 = 22.3$, $\sigma_3 = 17.9$.

Eady model: $t_{opt} = 48$ h, $t = 42$ h

streamfunction perturbation, SV 1 (top), SV 2 (middle), SV 3 (bottom)



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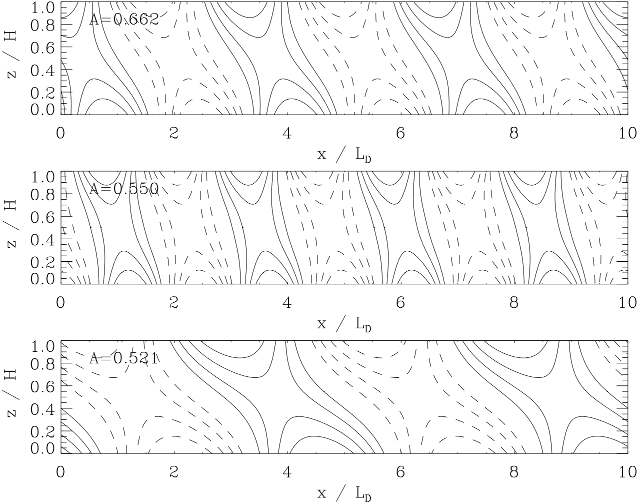

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Growth mechanisms

- PV unshielding
- intensification of boundary thermal anomalies through winds associated with interior PV anomalies
- interaction of waves on upper and lower boundary

see Badger and Hoskins (2001); Morgan and Chen (2002); DeVries and Opsteegh (2005); De Vries et al. (2009)

Singular vectors in the operational EPS

- $t_{\mathrm{opt}} \equiv t_1 - t_0 = 48$ h
- resolution: T42L62
- **Extra-tropics:** 50 SVs for N.-Hem. (30°N–90°N) + 50 SVs for S.-Hem.(30°S–90°S). Tangent-linear model with vertical diffusion and surface friction only.

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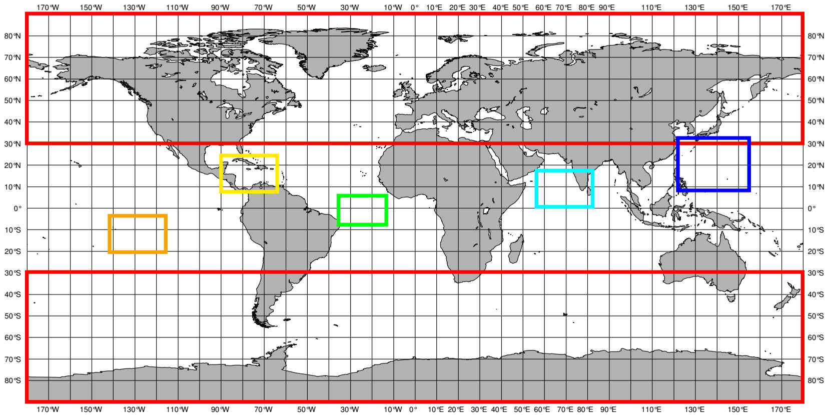
- **Localisation** is required to avoid that too many leading singular vectors are located in the dynamically more active winter hemisphere (Buizza 1994). Also required to obtain (more slowly growing) perturbations associated with tropical cyclones (Puri et al. 2001). In order to optimise perturbations for a specific region simply replace the propagator **M** in the equations by **PM**, where **P** denotes the projection operator which sets the state vector ($T, u, v, \ln p_{\text{sfc}}$ in grid-point space) to zero outside the region of interest and is the identity inside it.

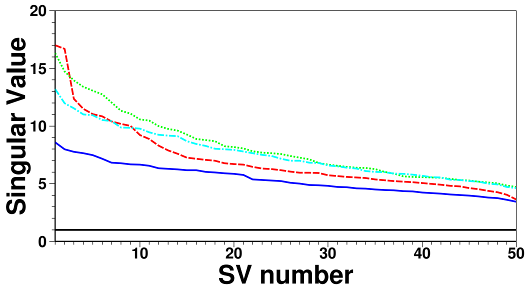
Schematic Opt. Areas



Singular values σ_j — extra-tropics

Northern Hem.

solid:	2005070100
dashed:	2005092100
dotted:	2005122100
chain-dashed:	2006032100



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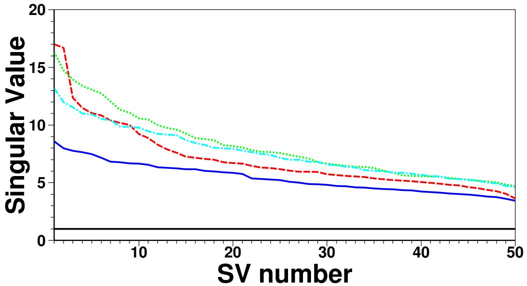

Southern Hem.

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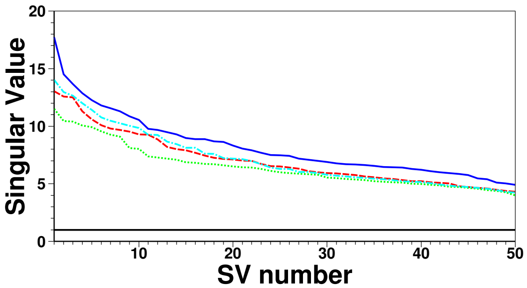
dotted: 2005122100
chain-dashed: 2006032100



Singular vector growth characteristics

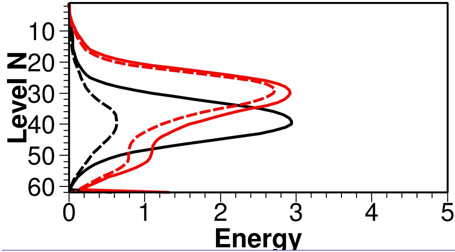
average energy of the leading 50 singular vectors

initial time (× 50), **final time t =48 h (× 1)**

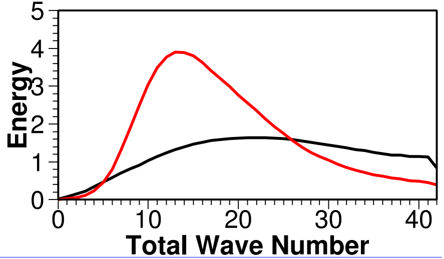
━━ : total energy; ━ ━ : kinetic energy

Northern hemisphere extra-tropics, 2006032100

vertical profile



spectrum



200 hPa ↔ level 20	300 hPa ↔ level 27
500 hPa ↔ level 35	700 hPa ↔ level 42
850 hPa ↔ level 48	925 hPa ↔ level 52

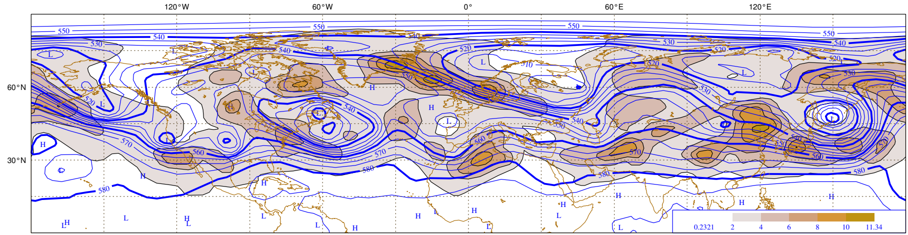
wave number	wave length
5	8000 km
10	4000 km
20	2000 km
40	1000 km

Regional distribution of Northern Hem. SVs

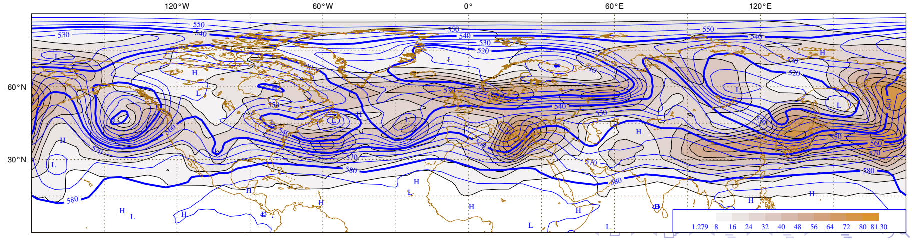
square root of vertically integrated total energy of SV 1–50 (shading)

500 hPa geopotential (contours)

initial singular vectors, 21 March 2006, 00 UTC



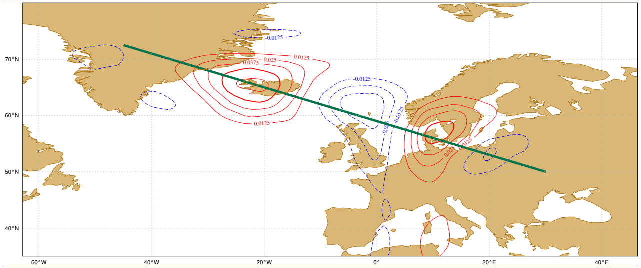
evolved singular vectors, 23 March 2006, 00 UTC



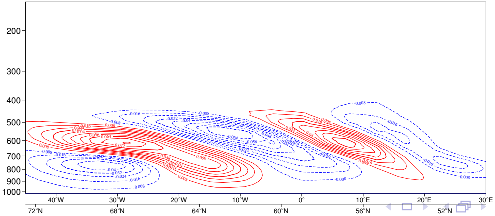
Singular vector 5: initial time

21 March 2006, 00 UTC

Temperature at ≈ 700 hPa



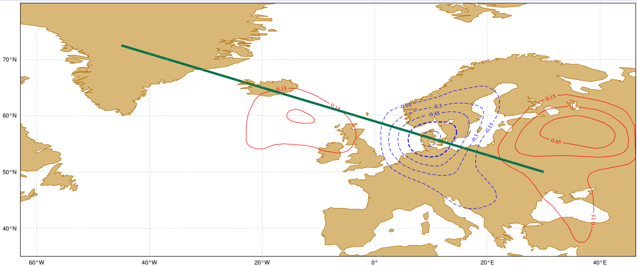
Cross section of temp 20060321 00 step 0 Expver 0001



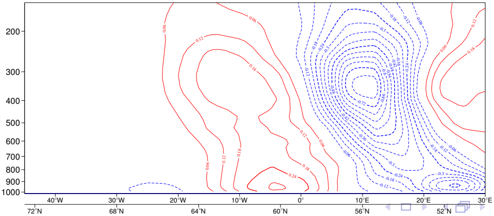
Singular vector 5: final time

23 March 2006, 00 UTC

meridional wind component at ≈ 300 hPa



Cross section of v-vel 20060321 00 step 48 Expver 0001



Initial condition perturbations

- Initial condition uncertainty is represented by a (multi-variate) Gaussian distribution in the space spanned by the leading singular vectors
- The perturbations based on a set of singular vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are of the form

$$\mathbf{x}_j = \sum_{k=1}^{m} \alpha_{jk} \mathbf{v}_k \tag{3}$$

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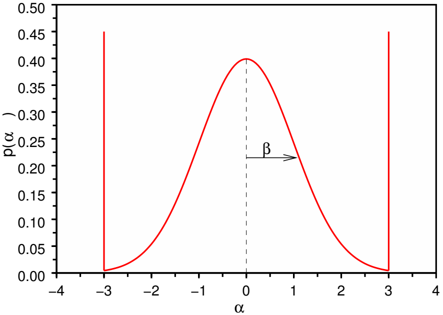
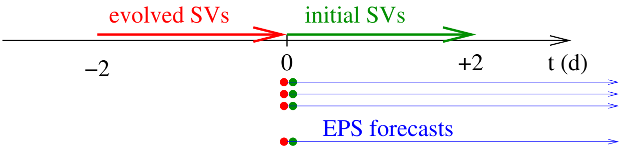
- The α_{jk} are independent draws from a truncated **Gaussian distribution**.

- The Gaussian is truncated at ± 3 standard deviations to avoid numerical instabilities for extreme values.
- The width of the distribution is set so that the spread of the ensemble matches the root-mean square error in an average over many cases ($\beta \approx 10$).



Initial condition perturbations (2)

- For the **extra-tropical perturbations**, the leading 50 initial singular vectors (in each hemisphere) are combined with the leading 50 evolved singular vectors (replaced by EDA perturbations since 22 June 2010)



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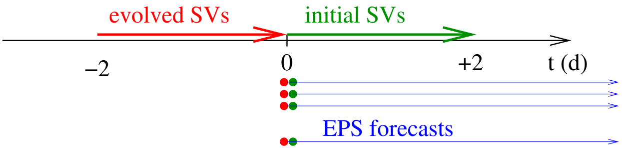

- For each of the (up to 6) optimisation regions **targeted on a tropical cyclone**, the leading 5 initial singular vectors are combined.

Initial condition perturbations (2)

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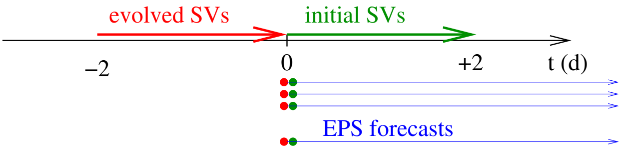


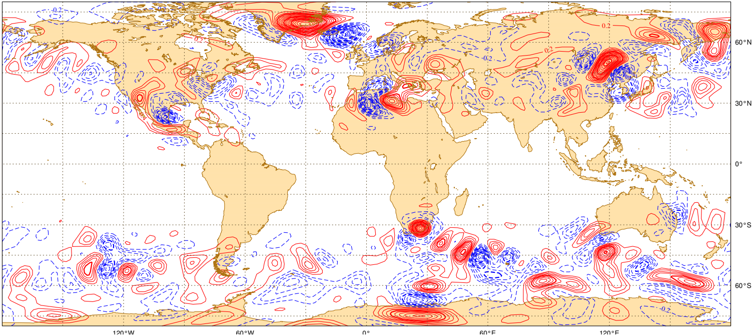
- For each of the (up to 6) optimisation regions **targeted on a tropical cyclone**, the leading 5 initial singular vectors are combined.
- To make sure that the ensemble mean is centred on the unperturbed analysis a **plus-minus symmetry** has been introduced:
  - coefficients for members 1, 3, 5, . . . , 49 are sampled,
  - the perturbation for members 2, 4, 6, . . . 50 is set to minus the perturbation of the member $j-1$ ($\mathbf{x}_j = -\mathbf{x}_{j-1}$).

Note: The sign of a singular vector itself is arbitrary.

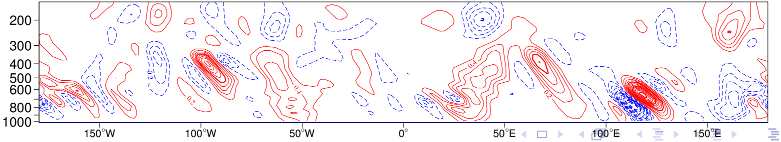
Initial condition perturbation for member 1

Temperature (every 0.2 K); 21 March 2006, 00 UTC

at ≈ 700 hPa



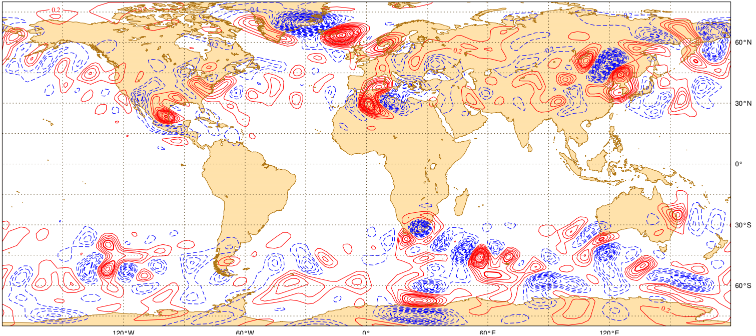
at 50°N



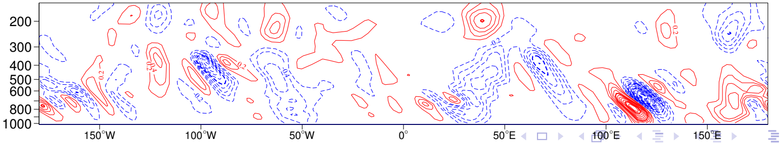
Initial condition perturbation for member 2

Temperature (every 0.2 K); 21 March 2006, 00 UTC

at ≈ 700 hPa



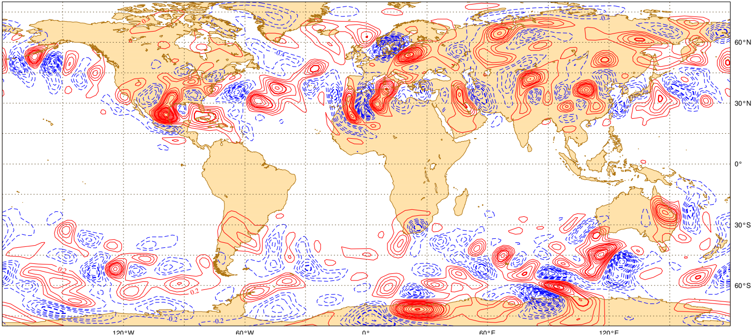
at 50°N



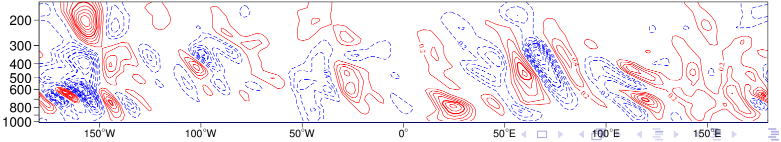
Initial condition perturbation for member 5

Temperature (every 0.2 K); 21 March 2006, 00 UTC

at ≈ 700 hPa



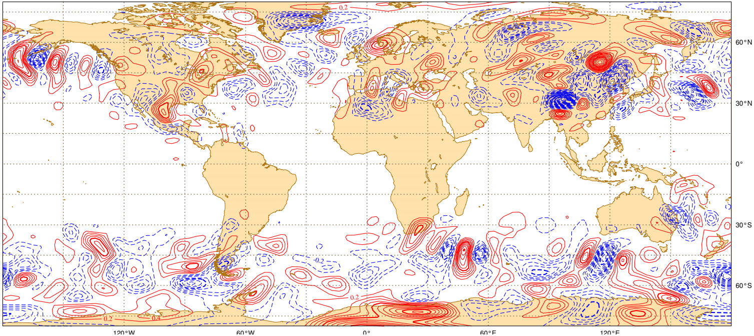
at 50°N



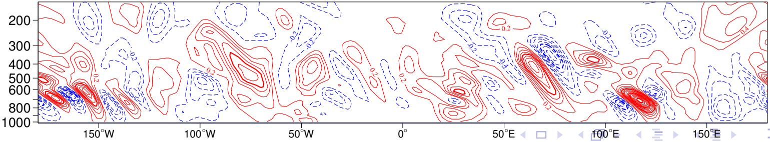
Initial condition perturbation for member 50

Temperature (every 0.2 K); 21 March 2006, 00 UTC

at ≈ 700 hPa



at 50°N



Appendix

linear algebra

m**-by-**n **matrix,** m rows and n columns $\mathbf{Q} = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$

matrix multiplication: entry in i-th row and j-th column is the inner product of row i of $\mathbf{A}$ and col. j of $\mathbf{B}$

$$\mathbf{C} = \mathbf{AB} \quad \Leftrightarrow \quad c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$$

matrix transpose: swap rows with columns

$$\mathbf{C} = \mathbf{A}^{\mathrm{T}} \quad \Leftrightarrow \quad c_{ij} = a_{ji} \quad ; \quad (\mathbf{A}\,\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}}$$

column vector: m-by-1 matrix

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

row vector: 1-by-m matrix

$$\mathbf{v}^{\mathrm{T}} = (v_1 \; \ldots \; v_m)$$

linear algrebra (cont.)

inner product:

$$\mathbf{x}^{\mathrm{T}}\mathbf{y} = \sum_{j=1}^{m} x_j y_j \quad \text{or more generally} \quad \mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{y} = \sum_{j=1}^{m}\sum_{k=1}^{m} x_j C_{jk} y_k$$

where **C** is symmetric ($\mathbf{C}^{\mathrm{T}} = \mathbf{C}$) and positive definite ($\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} > 0$ for $\mathbf{x} \neq 0$).

orthogonal and orthonormal sets of vectors:

orthogonal: $\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0$

orthonormal = orthogonal and normalised: $\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_k = \delta_{jk} = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k \end{cases}$

orthogonal matrix: row vectors and column vectors are orthonormal sets of vectors

$$\mathbf{V}^{\mathrm{T}}\mathbf{V} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Tangent-linear system

Consider the spatially discretised equations describing the atmospheric dynamics and physics written in this form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = F(\mathbf{x}), \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^N$ denotes the N-dimensional state vector and $F(\mathbf{x}) \in \mathbb{R}^N$ its tendency.

Let $\mathbf{x}_r(t)$ be a solution of (4). Then the *tangent-linear system* is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}(\mathbf{x}_r(t))\,\mathbf{x}, \tag{5}$$

where $[\mathbf{A}(\mathbf{x})]_{jk} = (\partial F_j/\partial x_k)(\mathbf{x})$ denotes the Jacobi matrix of F.

For any solution $\mathbf{x}$ of (5), $\mathbf{x}_r + \varepsilon\mathbf{x}$ approximates a solution of (4) starting at $\mathbf{x}_r(t_0) + \varepsilon\mathbf{x}(t_0)$ to first order in ε.

The *(tangent-linear) propagator* from t_0 to t_1 is the matrix $\mathbf{M}(t_0, t_1)$ such that $t \mapsto \mathbf{M}(t_0, t)\mathbf{x}_0$ is a solution of (5) for any initial perturbation $\mathbf{x}_0$ and where $\mathbf{M}(t_0, t_0) = \mathbf{I}$.

Singular vectors of the propagator

$$\text{Consider an initial time norm} \quad \|\mathbf{x}\|_i^2 = \mathbf{x}^{\mathrm{T}}\mathbf{C}_0^{-1}\mathbf{x}$$
$$\text{and a final time norm} \quad \|\mathbf{x}\|_f^2 = \mathbf{x}^{\mathrm{T}}\mathbf{D}\mathbf{x},$$

where $\mathbf{C}_0$ and $\mathbf{D}$ are positive definite symmetric matrices. Then, we consider the propagator for a fixed time interval $\mathbf{M} \equiv \mathbf{M}(t_0, t_1)$ and apply scalings depending on the norms (non-dimensionalisation). The reason for this particular scaling of the propagator will become obvious later.
The singular value decomposition of the scaled propagator $\widetilde{\mathbf{M}}$ is

$$\widetilde{\mathbf{M}} \equiv \mathbf{D}^{1/2}\mathbf{M}\mathbf{C}_0^{1/2} = \widetilde{\mathbf{U}}\mathbf{S}\widetilde{\mathbf{V}}^{\mathrm{T}} \qquad (6)$$

Here, $\mathbf{S}$ is the diagonal matrix containing the decreasing singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N$; $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$ contain the non-dimensional left and right singular vectors, respectively (see slide 6).

Singular vectors of the propagator (2)

Usually, the left and right singular vectors are transformed to physical space:

$$\begin{array}{rl} \text{initial SVs} & \mathbf{V} = \mathbf{C}_0^{1/2}\widetilde{\mathbf{V}} \\ \text{normalised evolved SVs} & \mathbf{U} = \mathbf{D}^{-1/2}\widetilde{\mathbf{U}} \end{array}$$

The initial SVs are orthonormal with respect to $\mathbf{C}_0^{-1}$: $\mathbf{v}_j^{\mathrm{T}}\mathbf{C}_0^{-1}\mathbf{v}_k = \delta_{jk}$.
The normalised evolved SVs are orthonormal with respect to $\mathbf{D}$: $\mathbf{u}_j^{\mathrm{T}}\mathbf{D}\mathbf{u}_k = \delta_{jk}$.

The singular value decomposition of the propagator in dimensional form reads

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}\mathbf{C}_0^{-1}. \tag{7}$$

Let $\mathbf{v}_j$ denote the j-th column of $\mathbf{V}$. It is referred to as (initial) singular vector j. From (7) it is obvious that

$$\mathbf{M}\mathbf{v}_j = \underbrace{\sigma_j\mathbf{u}_j}_{\text{evolved SV}} . \tag{8}$$

EOFs of a linear estimate of the FC error covariance matrix

The initial singular vectors are orthonormal with respect to the estimate of the inverse initial error covariance matrix:

$$\mathbf{V}^{\mathrm{T}}\mathbf{C}_0^{-1}\mathbf{V} = \mathbf{I} \quad \text{(identity matrix);} \qquad \Rightarrow \quad \mathbf{C}_0 = \mathbf{V}\mathbf{V}^{\mathrm{T}}.$$

Using (7), the linear evolution of the covariance estimate from t_0 to t_1 can be expressed as

$$\mathbf{C}_1 = \mathbf{M}\mathbf{C}_0\mathbf{M}^{\mathrm{T}} = \mathbf{U}\mathbf{S}^2\mathbf{U}^{\mathrm{T}}. \tag{9}$$

with scaling of errors with $\mathbf{D}^{1/2}$:

$$\mathbf{D}^{1/2}\mathbf{C}_1\mathbf{D}^{1/2} = \mathbf{D}^{1/2}\mathbf{M}\mathbf{V}\mathbf{V}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{D}^{1/2} = (\mathbf{D}^{1/2}\mathbf{U})\mathbf{S}^2(\mathbf{D}^{1/2}\mathbf{U})^{\mathrm{T}}. \tag{10}$$

- Eqn. (10) provides the EOF decomposition of the (scaled) forecast error covariance estimate.
- The leading singular vectors evolve into the directions of the leading EOFs of the (scaled) forecast error covariance estimate.
- This property makes the singular vectors an attractive basis for sampling initial condition uncertainties.

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Tangent-linear and adjoint models

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