Directional spread parameter in ERA5:

My starting point is the book by Leo Holthuijsen "Waves in Oceanic and Coastal Waters" (ISBN: 9780521129954), section 6.3.4 reproduced below (I strongly recommend this book).

In formula (6.3.21) of that book, there is an implicit reference that the angle θ is relative to the mean wave direction at that frequency (**Θ**(f)). To be more general, it should be re-written as

$$\sigma_\theta^2(f) = \int_{-\pi}^{\pi} \left[2 \, \sin\left(\frac{1}{2}\left(\theta - \boldsymbol{\Theta}(f)\right)\right)\right]^2 D(\theta) d\theta$$

(1)

where by definition $D(\theta) = \frac{F(f,\theta)}{E(f)}$

with $E(f) = \int_{-\pi}^{\pi} F(f,\theta) \, d\theta$

Noting at this point that $\sigma_\theta(f)$ is the one sided directional width for a given frequency (f).

It is expressed in **radian**.

In (1), the mean wave direction **Θ**(f) is computed as follows: $\boldsymbol{\Theta}(f) = \ atan\left\{\frac{\int_{-\pi}^{\pi} \sin\theta \ F(f,\theta) d\theta}{\int_{-\pi}^{\pi} \cos\theta \ F(f,\theta) d\theta}\right\}$

On the other hand, in the ECMWF IFS documentation part VII, the directional spread is defined in (10.9) such that

$$\sigma_\theta^2 = 2\left(1 - \frac{I_1}{E_0}\right)$$

(2)

With

$$I_1 = \iint_{-\pi}^{\pi} \cos\left(\theta - \boldsymbol{\Theta}(f)\right) F(f,\theta) d\theta df$$

(3)

And

$$E_0 = \iint_{-\pi}^{\pi} F(f,\theta) \ d\theta df$$

(4)

Using the following relation

$$\cos a = 1 - 2\left[\sin\left(\frac{a}{2}\right)\right]^2$$

One can rewrite (2) using (3) as follows

$$\sigma_\theta^2 = \iint_{-\pi}^{\pi} \left[2\,\sin\left(\frac{1}{2}\left(\theta - \boldsymbol{\Theta}(f)\right)\right)\right]^2 \frac{F(f,\theta)}{E_0}\, d\theta df$$

Or

$$\sigma_\theta^2 = \frac{1}{E_0} \iint_{-\pi}^{\pi} E(f) \left[2\,\sin\left(\frac{1}{2}\left(\theta - \boldsymbol{\Theta}(f)\right)\right)\right]^2 \frac{F(f,\theta)}{E(f)}\, d\theta df$$

(5)

Finally, rewriting (5)

$$\sigma_\theta^2 = \frac{1}{E_0} \int \boldsymbol{E(f)} \left\{\int_{-\pi}^{\pi} \left[2\,\sin\left(\frac{1}{2}\left(\theta - \boldsymbol{\Theta}(f)\right)\right)\right]^2 \frac{F(f,\theta)}{E(f)}\, d\theta\right\} \boldsymbol{df}$$

(6)

(6) can be interpreted as the weighted average in frequency space of (1), which is the definition of the directional spread used in ERA5 for the total sea (one number describing the mean directional spreading). As noted in (1), its unit is **radian**.

energy scale parameter α is equally a function of dimensionless fetch $\tilde{F}$ but it can also be expressed in terms of the dimensionless peak frequency, e.g., from Lewis and Allos (1990):

$$\alpha = 0.0317\tilde{f}_{peak}^{0.67} \tag{6.3.17}$$

In JONSWAP, the scatter in the values of the shape parameters γ, σ_a and σ_b was so large that no dependence on the dimensionless fetch could be discerned. The average values were $\gamma = 3.3$, $\sigma_a = 0.07$ and $\sigma_b = 0.09$. Others have repeated the JONSWAP study at different times and locations with essentially the same results. The transition to fully developed sea states is apparently poorly defined but, if required, it can be obtained with (Lewis and Allos, 1990; see also Eqs. 8.3.10 and 8.3.11)

$$\begin{aligned} \gamma &= 5.870\tilde{f}_{peak}^{0.86} \\ \sigma_a &= 0.0547\tilde{f}_{peak}^{0.32} \\ \sigma_b &= 0.0783\tilde{f}_{peak}^{0.16} \end{aligned} \tag{6.3.18}$$

These relationships are consistent with the JONSWAP observations but they have been forced to be equal to the values of $\alpha = 0.0081$ and $\gamma = 1.0$ for the Pierson and Moskowitz spectrum at $\tilde{f}_{peak} = 0.13$ (the values of σ_a and σ_b are irrelevant when $\gamma = 1.0$).

Literature:
Alves *et al.* (2003), Ewing and Laing (1987), Huang *et al.* (1981, 1990a), Mitsuyasu (1968, 1969), Mitsuyasu *et al.* (1980), Phillips (1985), Resio *et al.* (1999), Toba (1973, 1997).

6.3.4 The two-dimensional wave spectrum

The two-dimensional frequency–direction spectrum is difficult to observe, as noted in Chapter 2. Usually, only some *overall* directional characteristics are observed, notably the mean direction and the directional *width* of the spectrum (representing the degree of short-crestedness of the waves). This concept of directional width requires the introduction of the directional distribution $D(\theta; f)$. It is essentially the cross-section through the two-dimensional spectrum *at a given frequency*, normalised such that its integral over the directions is unity. In other words, it is a normalised, circular transect through the two-dimensional spectrum (see Fig. 6.7):

$$D(\theta; f) = \frac{E(f, \theta)}{E(f)} \tag{6.3.19}$$

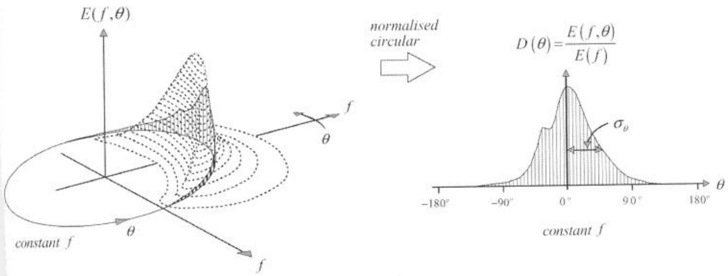

Figure 6.7 The directional energy distribution at a given frequency under arbitrary conditions and its (one-sided) directional width σ_θ.

That the integral over directions of this function is unity is readily shown as follows:

$$\int_0^{2\pi} D(\theta; f)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{E(f,\theta)}{E(f)}\mathrm{d}\theta = \frac{\int_0^{2\pi} E(f,\theta)\,\mathrm{d}\theta}{E(f)} = \frac{E(f)}{E(f)} = 1 \qquad (6.3.20)$$

Note that the directional distribution $D(\theta; f)$ gives the *normalised* distribution of the wave energy density over directions at *one* frequency, whereas the two-dimensional spectrum $E(f,\theta)$ gives the *non-normalised* distribution over both frequency and direction. Obviously $D(\theta; f)$ may vary with frequency. Very often such frequency-dependence is ignored in the notation, so that $D(\theta; f)$ is often written as $D(\theta) = D(\theta; f)$. Strictly speaking $D(\theta; f)$ is dimensionless, but it can be considered to have a *unit* of [1/angle], i.e., [1/rad] or [1/degree].

The directional spreading of the waves can be defined as the (one-sided) directional width of $D(\theta)$, denoted as σ_θ (see Fig. 6.7), in analogy with the conventional definition of standard deviation: $\sigma_\theta^2 = \int_{-\pi}^{+\pi} \theta^2 D(\theta)\mathrm{d}\theta$ (where θ is taken relative to the mean wave direction). However, for various reasons, it is better to replace θ with $\sin\theta$, or better still, by $2\sin\left(\frac{1}{2}\theta\right)$, so that

$$\sigma_\theta^2 = \int_{-\pi}^{+\pi} \left[2\sin\left(\tfrac{1}{2}\theta\right)\right]^2 D(\theta)\,\mathrm{d}\theta \qquad (6.3.21)$$

Young *et al.* (1996) and Ewans (1998) have published a large number of observations of σ_θ, which are summarised in Fig. 6.8, showing that σ_θ varies from approximately 30° at the peak frequency f_{peak} to about 60° at $3 f_{peak}$ (but the scatter in the observations is rather large). They report finding little or no dependence of

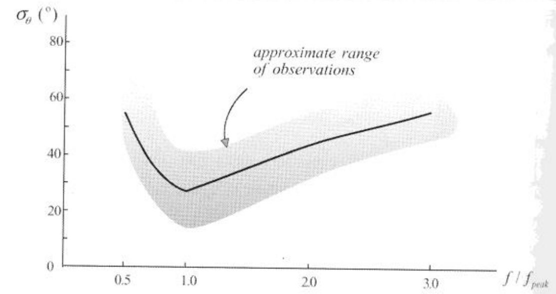

Figure 6.8 The directional width (one-sided) of the directional energy distribution $D(\theta; f)$ as a function of normalised frequency and the expression of Eq. 6.3.22. Observations of Young *et al.* (1996) and Ewans (1998).

σ_θ on wind speed. A reasonable fit to their observations is

$$\sigma_\theta = \begin{cases} 26.9(f/f_{peak})^{-1.05} & \text{in degrees, for } f < f_{peak} \\ 26.9(f/f_{peak})^{0.68} & \text{in degrees, for } f \geq f_{peak} \end{cases} \tag{6.3.22}$$

The *shape* of the distribution $D(\theta)$ is not well known, not even in the idealised situation that we consider here. It is usually speculated that this distribution has a maximum in the wind direction (most of the wave energy travelling downwind) and that it falls off gradually to the offwind directions (see Fig. 6.9, but see Note 6D). Several expressions with this character have been suggested to describe $D(\theta)$. The best-known and probably most widely used is the $\cos^2\theta$ model (e.g., Pierson *et al.*, 1952):

$$D(\theta) = \begin{cases} \dfrac{2}{\pi}\cos^2\theta & \text{for } |\theta| \leq 90^\circ \\ 0 & \text{for } |\theta| > 90^\circ \end{cases} \tag{6.3.23}$$

where the direction θ is taken relative to the mean wave direction. Its directional width $\sigma_\theta \approx 30^\circ$. As Eqs. (6.3.22) show, this value agrees well with observations near the peak frequency. Moreover, it is a constant, i.e., independent of wind and frequency, which is convenient for many engineering applications. To obtain more flexibility, this model has been generalised to the $\cos^m\theta$ model:

$$D(\theta) = \begin{cases} A_1 \cos^m\theta & \text{for } |\theta| \leq 90^\circ \\ 0 & \text{for } |\theta| > 90^\circ \end{cases} \tag{6.3.24}$$